

Example 8. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A .

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic equation is: $\det(A - \lambda I) = 0 \implies \begin{vmatrix} 0 - \lambda & -1 \\ 1 & 0 - \lambda \end{vmatrix} \implies \lambda^2 + 1 = 0 \implies \lambda = \pm \iota$.

Case 1. Let $\lambda = \iota$ and $X = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \iota \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} &= \begin{bmatrix} \iota x_1 \\ \iota x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -\iota x_1 - x_2 \\ x_1 - \iota x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For eigenvalue $\lambda = \iota$, we get $x_1 = \iota c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} \iota \\ 1 \end{bmatrix} c$; $c \in \mathbb{F}$.

Case 2. Let $\lambda = -\iota$ and $X = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -\iota \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} &= -\begin{bmatrix} \iota x_1 \\ \iota x_2 \end{bmatrix} \\ \implies \begin{bmatrix} \iota x_1 - x_2 \\ x_1 + \iota x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For eigenvalue $\lambda = -\iota$, we get $x_1 = -\iota c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} -\iota \\ 1 \end{bmatrix} c$; $c \in \mathbb{F}$.

1 Diagonalisation of a Matrix

Similar Matrices. Let us consider all $n \times n$ matrices over the field $\mathbb{F}$. An $n \times n$ matrix A is said to be similar to an $n \times n$ matrix B , if there exists a non-singular matrix P of order $n \times n$ such that $B = P^{-1}AP$. We say that A and B are similar matrices.

$$\begin{aligned} B &= P^{-1}AP \\ \implies PB &= PP^{-1}AP = (PP^{-1})AP = IAP = AP \\ \implies PBP^{-1} &= APP^{-1} = A \\ \implies A &= Q^{-1}BQ, \text{ where } Q = P^{-1}. \end{aligned}$$

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then, A is said to be Diagonalisable, if A is similar to an $n \times n$ diagonal matrix.

Example 1. Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Solution. *Step 1.* Let $\lambda \in \mathbb{F}$ be the eigenvalue. Then

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \Rightarrow \begin{vmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda &= -2, 4, -2. \end{aligned}$$

Step 2. Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the eigenvector.

$$\begin{aligned} AX &= \lambda X \\ \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ 3x_1 - 5x_2 + 3x_3 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} &= -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 3x_1 - 3x_2 + 3x_3 \\ 3x_1 - 3x_2 + 3x_3 \\ 6x_1 - 6x_2 + 6x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Which gives us $x_1 - x_2 + x_3 = 0$. So, $x_2 = c$, $x_3 = d$, $x_1 = c - d$, where $c, d \in \mathbb{F}$. The required solution: $X = (c - d, c, d)$. The eigenvector corresponding to $\lambda = -2$ is $(1, 1, 0)$ and $(-1, 0, 1)$, i.e.,

$$\left\{ \begin{bmatrix} c - d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} ; c, d \in \mathbb{F} \right\}.$$

Step 3. For $\lambda = 4$, follow the same process and find the eigenvector. it will be continued in the next lecture.

Example 2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are similar matrices.

Proposition 1. Let $A = (a_{ij})_{n \times n}$ with eigenvalues $d_1, d_2, \dots, d_n \in \mathbb{F}$, which are not necessarily distinct. Let $D = \text{diag}(d_1, d_2, \dots, d_n)$. Then, for any matrix P , $AP = PD$ holds if and only if the i th column vector of P be an eigenvalue of A corresponding to d_i .

Example 1. (continued...) Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

Solution. Step 3. For $\lambda = 4$, let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the eigenvector.

$$\begin{aligned} AX &= \lambda X \\ \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ 3x_1 - 5x_2 + 3x_3 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} &= 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} -3x_1 - 3x_2 + 3x_3 \\ 3x_1 - 9x_2 + 3x_3 \\ 6x_1 - 6x_2 + 0x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

From here we get $x_1 = x_2$, $x_1 + x_2 = x_3$ and $x_1 + x_3 = 3x_2$. If $x_1 = c = x_2$, where $c \in \mathbb{F}$, then $x_3 = 2c$. Therefore, the required eigenvector corresponding to the eigenvalue 4 is $c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $c \in \mathbb{F}$.

Step 4. The required matrix $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Now we need to check whether $\det(P)$ is 0 or not, i.e., whether the matrix P is non-singular or not. It is easy to find out that the determinant comes out to be a non-zero value.

Next, we check $D = P^{-1}AP$ holds or not, where $P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. We use the Proposition 1 and check $DP = AP$

Example 3. Diagonalise $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Solution. *Step 1.* Let $\lambda \in \mathbb{F}$ be the eigenvalue. Then

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \implies \begin{vmatrix} 1 - \lambda & 1 & -2 \\ -1 & 2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{vmatrix} &= 0 \\ \implies \lambda &= -1, 1, 2. \end{aligned}$$

Step 2. The eigenvectors corresponding to $\lambda = 1, 2, -1$ are $c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, respectively, where $c_1, c_2, c_3 \in \mathbb{F}$. Hence, the required matrix $P = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Finally, we consider

$$\begin{aligned} AP &= \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 0 \\ 1 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= PD, \quad \text{where } D = \text{diag}(1, 2, -1). \end{aligned}$$

1 Orthogonal Matrix

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then, A is said to be an Orthogonal Matrix, if $AA^T = I_n$.

Proposition 1. *If A is orthogonal, then A is non-singular.*

We can see $AA^T = I_n \implies |A||A^T| = 1 \implies |A|^2 = 1$. Hence, $\det(A) \neq 0$.

Proposition 2. *If A is orthogonal, then $A^{-1} = A^T$.*

Proposition 3. *If A is orthogonal, then A^{-1} is also orthogonal.*

2 Complex Matrix

A matrix A , whose elements are taken from $x \in \mathbb{C}$, is called a complex matrix. This means that A can be expressed as $P + \iota Q$, where P and Q are real matrices. Then, $A = P + \iota Q$, $\bar{A} = P - \iota Q$. Some properties of the complex matrices are given below.

- (i) $\bar{\bar{A}} = A$.
- (ii) A and B are two complex matrices, then $\overline{AB} = \bar{A}\bar{B}$.
- (iii) For a complex matrix, $(\bar{A})^T = \overline{A^T}$. The transpose of $\bar{A}$ is denoted by A° , $(\bar{A})^T = A^\circ$.

3 Hermitian and Skew-Hermitian Matrices

A complex matrix is said to be Hermitian Matrix if $A = A^\circ$, $A^\circ = (\bar{A})^T$. A complex matrix is said to be a Skew-Hermitian Matrix if $A = -A^\circ$, $A^\circ = -(\bar{A})^T$.

Example 1. Check the matrix $A = \begin{bmatrix} 1 & 2 & 3\iota \\ 2 & 0 & 4 \\ -3\iota & 4 & 2 \end{bmatrix}$ is Hermitian or Skew-Hermitian.

Solution. We get $\bar{A} = \begin{bmatrix} 1 & 2 & -3\iota \\ 2 & 0 & 4 \\ 3\iota & 4 & 2 \end{bmatrix}$ and then $(\bar{A})^T = \begin{bmatrix} 1 & 2 & 3\iota \\ 2 & 0 & 4 \\ -3\iota & 4 & 2 \end{bmatrix} = A$. Thus, A is a Hermitian matrix.

Example 2. Check the matrix $A = \begin{bmatrix} 0 & 2 & \iota \\ -2 & 0 & 1 + \iota \\ \iota & -1 + \iota & 0 \end{bmatrix}$ is Hermitian or Skew-Hermitian.

Solution. We get $\bar{A} = \begin{bmatrix} 0 & 2 & -\iota \\ -2 & 0 & 1 - \iota \\ -\iota & -1 - \iota & 0 \end{bmatrix}$ and then $(\bar{A})^T = \begin{bmatrix} 0 & -2 & -\iota \\ 2 & 0 & -1 - \iota \\ -\iota & 1 - \iota & 0 \end{bmatrix} = -A$. Thus, A is a Skew-Hermitian matrix.

4 Unitary Matrix

A matrix A is said to be a Unitary Matrix if $A\bar{A}^T = I$.

Example 1. Let $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where $\omega = \frac{-1 \pm \sqrt{3}\iota}{2}$ and the relations $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$ hold. Check whether A is Unitary matrix or not.

Theorem 1. If $H = P + \iota Q$ be a Hermitian matrix, then

- (i) the diagonal elements of H are all real numbers.
- (ii) P is a real Symmetric matrix and Q is a real Skew-Symmetric matrix.

Theorem 2. If $S = M + \iota N$ be a Skew-Hermitian matrix, then

- (i) the diagonal elements of S are either 0 or imaginary numbers.
- (ii) M is a real Skew-Symmetric matrix and N is a real Symmetric matrix.

NPTTEL

1 Equivalent Matrices and Elementary Matrices

An $n \times n$ matrix B is said to be equivalent to an $n \times n$ matrix A over the same field $\mathbb{F}$ if B can be obtained from A by a finite number of elementary rows and columns operations.

Definition. For a given $m \times n$ matrix A , a row-reduced echelon matrix B can be obtained by applying on A a finite number of elementary row operation on A . On B , we can apply column operations to reduce it into a column reduced echelon matrix, say C . This C has the following properties:

- (a) No zero row is followed by a non-zero row.
- (b) No zero column is followed by a non-zero column.
- (c) The leading 1 in each non-zero row is the only non-zero element.
- (d) The leading 1 in each non-zero column is the only non-zero element.
- (e) The leading 1 in the k th row is the leading 1 in the k th column.

Thus, C has the form $C = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$, where I is the identity matrix of order r , O_{pq} is the zero matrix of order $p \times q$. Then, C is called the Fully-Reduced-Normal-Form.

Example 1. Obtain the fully reduced normal form of the matrix A give by $A = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix}$.

Solution. Let us first apply the elementary row operations.

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix} \xrightarrow{\substack{R_4 - 3R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\
 &\xrightarrow{\substack{R_3 - 2R_2 \\ R_1 - R_2, R_4 - R_2}} \begin{bmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_3 \\ R_1 + 2R_3}} \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 &= B \rightarrow \text{row equivalent to } A \\
 &\xrightarrow{\substack{c_5 - 2c_1 \\ c_2 - 3c_1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{c_{23} \\ c_{34}}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 &= C = \begin{bmatrix} I_3 & O_{3,2} \\ O_{1,3} & O_{1,2} \end{bmatrix} \rightarrow \text{fully reduced normal form.}
 \end{aligned}$$

Example 2. Show that the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix}$ is non-singular and express it as the product of elementary matrices.

Solution. We start the process and apply operation as follows:

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix} \xrightarrow[\frac{R_3-6R_1}{R_2-3R_1}]{} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & -\frac{3}{2} \\ 0 & 2 & 0 \end{bmatrix} \\ &\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\frac{R_1-\frac{1}{2}R_3}{R_2+\frac{1}{2}R_3}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \end{aligned}$$

Clearly, A is row-equivalent to I_3 . Since, $|I_3| \neq 0$, this implies A is non-singular. In short,

$$\begin{aligned} I_3 &= \left[\left(R_2 + \frac{1}{2}R_3 \right) \left(R_1 - \frac{1}{2}R_3 \right) (R_3 - 2R_2) \left(\frac{1}{3}R_2 \right) (R_2 - 3R_1) (R_3 - 6R_1) \left(\frac{1}{2}R_1 \right) \right] A \\ &= \left[E_{23} \left(\frac{1}{2} \right) E_{13} \left(-\frac{1}{2} \right) E_{32} (-2) E_2 \left(\frac{1}{3} \right) E_{21} (-3) E_{31} (-6) E_1 \left(\frac{1}{2} \right) \right] A \\ &= \left[\left(R_2 + \frac{1}{2}R_3 \right) \left(R_1 - \frac{1}{2}R_3 \right) (R_3 - 2R_2) \left(\frac{1}{3}R_2 \right) (R_2 - 3R_1) (R_3 - 6R_1) \left(\frac{1}{2}R_1 \right) \right] A \end{aligned}$$

$$\begin{aligned} A &= \left[E_{23} \left(\frac{1}{2} \right) E_{13} \left(-\frac{1}{2} \right) E_{32} (-2) E_2 \left(\frac{1}{3} \right) E_{21} (-3) E_{31} (-6) E_1 \left(\frac{1}{2} \right) \right]^{-1} I_3 \\ &= \left\{ E_1 \left(\frac{1}{2} \right) \right\}^{-1} \{ E_{31} (-6) \}^{-1} \{ E_{21} (-3) \}^{-1} \left\{ E_2 \left(\frac{1}{3} \right) \right\}^{-1} \{ E_{32} (-2) \}^{-1} \left\{ E_{13} \left(-\frac{1}{2} \right) \right\}^{-1} \left\{ E_{23} \left(\frac{1}{2} \right) \right\}^{-1} I \\ &= \left[E_1 (2) E_{31} (6) E_{21} (3) E_2 (3) E_{32} (2) E_{13} \left(\frac{1}{2} \right) E_{23} \left(-\frac{1}{2} \right) \right] I. \end{aligned}$$

Definition. An $n \times n$ matrix obtained by applying a single elementary row operations on I_n is said to be an Elementary Matrix of order n . There are three types of elementary matrices.

1. The elementary matrix obtained by applying R_{ij} on I_n is denoted by E_{ij} .
2. The elementary matrix obtained by applying cR_{ij} on I_n is denoted by $E_i(c)$.
3. The elementary matrix obtained by applying $R_i + cR_j$ on I_n is denoted by $E_{ij}(c)$.

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{13}(c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$